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J. U. Beusch

The Effect of Pseudorandom Frequency Hopping on the Probability of Simultaneous Usage of a Communication Satellite

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY

LINCOLN LABORATORY

THE EFFECT OF PSEUDORANDOM FREQUENCY HOPPING ON THE PROBABILITY OF SIMULTANEOUS USAGE OF A COMMUNICATION SATELLITE

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ABSTRACT

When the up-link frequency band repeated by a communications satellite is hopped in a periodic manner, when users are divided so that a group of them can transmit only when a particular frequency band is repeated, and when message arrivals for one user are independent of message arrivals for any other user, the users in a group are independent. When the frequency hopping is done in a pseudorandom rather than a periodic manner, the users are dependent. The effect of the dependence is, in practical cases, to increase the probability of there being a large number of simultaneous users which increases the probability of system overload. General expressions for the probability of system overload and for the fraction of the information lost due to overload are obtained. These expressions illustrate the effect of the dependence induced by the pseudorandom frequency hopping. Examples are presented.

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The Effect of Pseudorandom Frequency Hopping on the Probability of Simultaneous Usage of a Communication Satellite

An Anti-Jam, Multiple Access Communications Satellite System

To develop effective resistance to up-link jamming, it is desirable to incorporate a form of signal processing, frequency hopping, in a communications satellite repeater. In a proposed frequency hopping satellite the repeater and all users are synchronized by means of a clock in the satellite. At the discrete times ..., $-2T$, 0, T, $2T$,... the frequency band repeated by the satellite is selected from one of N possible bands where N is a power of 2. The selection is made by using the digits of a pseudorandom sequence which is generated by the satellite and by each of the users. The basic anti-jam capability of this system arises from the fact that each user can determine, from his copy of the pseudorandom sequence, what frequency band the satellite will repeat during succeeding time intervals while a hostile jammer cannot.

In a communications satellite system, users communicate in pairs since one must be receiving when another transmits. If any user can communicate with any other user, message flow between one pair of users can interact with message flow between another pair if the same user is a member of both pairs. On the other hand, if each user can report back only to a central station and this central station can communicate with many users simultaneously, this interaction between pairs does not occur. This report analyzes the effect upon system behavior of frequency hopping in the satellite and all other forms of interaction between pairs of users is ignored.

An attractive method of system organization for tactical communications when there are a large number of user pairs, is to divide the pairs into N groups and to assign one of the N satellite repeater bands to each group. Then all user pairs in a common group will transmit-receive only when the satellite repeats their frequency*. The users will not have to frequency hop. This reduces the complexity of the equipment that each user must have. If there are a large number of users (e.g., 1000) this will represent a considerable saving. The possibility of interference between groups is also eliminated.

The mathematical model to be described in the next section represents a good approximation to a realistic tactical situation.

Mathematical Model

Consider a large number of user pairs which are divided into N groups where group j, $1 \le j \le N$, contains M_j pairs. At the discrete times ..., $-2T$, $-T, 0, T, 2T, \ldots$ a decision is made as to which group has exclusive use of a communications satellite repeater for the next T seconds. The decision is made at random and the probability that group j uses the repeater is P_j. The outcome of the decision at a given time is independent of the outcomes of all previous decisions.

At the discrete times ...., $-2T, -T, 0, T, 2T, \ldots$ a message may arrive for any of the pairs to transmit-receive providing that pair is idle. A

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* Actually the user pairs can be divided into N/k groups where $k = 1, 2, \ldots, N/2$ and each group transmits when the satellite repeats any one of k frequency bands. The results of this report also apply to this case.

pair is considered to be idle at a given time if, in the event that no message arrives at that time, the pair will have no part of a message either being transmitted or waiting to be transmitted during the succeeding time interval of length T. The occurrence of a message arrival for a user pair is independent of the occurrence of an arrival for any other pair at the same time. The occurrence of an arrival for a user pair is independent of the occurrence of an arrival for any pair at any previous time. For each pair in group j, $1 \le j \le N$, the probability that a message arrival occurs at any particular time when the pair is idle is p_j. The occurrence of a message arrival at a discrete time in the model represents an arrival in the interval of length T immediately preceding that time in the physical process being modeled.

User pair i of group j, $1 \le i \le M_j$, $1 \le j \le N$, can transmit at an instantaneous rate of r_{ij} bits per second when group j transmits,and each message that arrives contains b_{ij} bits. The ratio b_{ij}/r_{ij} which is the actual transmission time for a message is an integer multiple of T and is assumed to be the same for all users in group j (i. e., $b_{ij}/r_{ij} = L_j T$ for all i where L_j is an integer). Where the meaning is clear the expression "transmit-receive" is abbreviated to "transmit" in this report.

Succeeding sections focus on one of the groups and the subscript j on the variables P_j, p_j, and L_j is omitted for convenience. The results apply to any group provided the proper values of these variables are used.

The Probability of the Occurrence of a System Overload

Given that the satellite is repeating the frequency band of the group being considered,let the probability that k of the m user pairs are simultaneously transmitting (active) be $P(k, m)$ and the probability that k or more of the m pairs are simultaneously transmitting be

$$Q(k, m) = \sum_{i=k}^{m} P(i, m) \quad . \tag{1}$$

The quantity, $Q(k + 1, m)$, is a measure of system performance in two ways.

1. If a large facility is paired with m' small stations, if each pair is in the same group, and if the large facility can only accommodate k simultaneous transmissions, the probability that the large facility becomes overloaded is $Q(k + 1, m')$.

2. In a hard-limiting satellite which is simultaneously repeating the transmissions of i users who have equal power the satellite output power per user $(\text{Power})_u$ is, to a good approximation,

$$(\text{Power})_u = f \ (\text{Power})_t / i$$

   where $(\text{Power})_t$ is the total satellite transmitted power and $1 - f$ is the fraction of the power lost due to repeater non-linearities (e.g., $f = 7/8$ in Reference 1). When the power per user falls below a certain threshold ($(\text{Power})_u \leq (\text{Power})_{min}$), performance of the communications links are degraded sharply. Therefore, when the number of simultaneous user exceeds some number, k, performance is degraded sharply and the quantity, $Q(k + 1, m)$ is again of interest. For a large number of users this problem cannot be circumvented by merely increasing $(\text{Power})_t$ since the transmitted power is severely limited by the weight, size, and configuration of the satellite in general and the solar cells in particular.

Fraction of the Information that is Lost Due to Overload

In a hard-limiting satellite when the number of equal power simultaneous active users exceeds k the down-link satellite power becomes insufficient to support the data rate in the communications links between user pairs. It is assumed that performance degradation is sufficiently sharp that, when the number of active users is less than or equal to k, messages are received and decoded with negligible error and when the number of active users exceeds k, messages are decoded with so many errors that they are worthless.

When the users of the group have equal power they will all be transmitting at the same data rate r bits per second. During a time interval in which the group transmits the average amount of information to be transmitted is

$$I_t = \sum_{i=0}^{m} irTP(i, m) \text{ bits}$$

since each active user pair transmits rT bits. If the number of active pairs exceeds k, no useful information is received so the average amount of information received is

$$I_r = \sum_{i=0}^{k} irTP(i, m) \text{ bits} \quad .$$

The fraction of the information that is lost is

$$F_L(k+1, m) = (I_t - I_r)/I_t = \left[\sum_{i=k+1}^{m} iP(i, m) \right] [rT/I_t]$$

To evaluate $F_L(k+1, m)$ and $Q(k+1, m)$ an expression for $P(i, m)$, the probability that i of the m user pairs in a group are simultaneously transmitting, is needed. This expression is derived in the following two sections.

The Probability Distribution of the Time Between Transmissions of a Group

The user pairs of a particular group may transmit during any interval of length T that the satellite repeats the frequency band of the group. Let q_h equal the probability that successive intervals during which pairs of a group can transmit are separated by h intervals of length T. Then

$$q_h = P(1-P)^h \qquad h \geq 0 \qquad . \tag{2}$$

Equation (2) follows from the fact that in order for the time between transmissions to be equal to hT, the satellite must make h successive decisions to repeat the frequency band of some other group followed by one decision to repeat the frequency band of the particular group.

The Probability Distribution of the Number of Simultaneous Users

Given that the satellite is repeating the frequency band of the group being considered, the probability that i of the m user pairs in the group are simultaneously active is defined to be $P(i, m)$. An expression for $P(i, m)$ is derived in Appendix A. The expression involves p, L, and q_h which are the probability that at a particular time a message arrives to be transmitted by a particular pair that is idle, the number of times a pair must transmit to completely send a message, and the probability that successive intervals during which pairs of the group can transmit are separated by h time intervals.

The expression is

$$P(i, m) = \binom{m}{i} \sum_{h=0}^{\infty} q_h [L - L(1-p)^{h+1}]^i \, [(1-p)^{h+1}]^{m-i} / [L - (L-1)(1-p)^{h+1}]^m \tag{3}$$

where

$$\binom{m}{i} = \frac{m!}{i!(m-i)!} \quad .$$

When decisions as to which group transmits are made in a pseudorandom manner, q_h is given by Eq. (2). With this form of q_h, the sum in Eq. (1) can be evaluated in closed form only for $L = 1$. For the special cases where $M = 1$ and 2, substitution of Eq. (1) into (2) for the case $L = 1$ yields

$$P(0, 1) = 1 - \alpha_1$$

$$P(1, 1) = \alpha_1$$

$$P(0, 2) = (1 - \alpha_1)^2 \{ P(1 - p + \tfrac{p}{P})^2 / [1 - (1 - P)(1 - p)^2] \}$$

$$P(1, 2) = 2\alpha_1 (1 - \alpha_1) \{ [1 - (1 - P)(1 - p)] / [1 - (1 - P)(1 - p)^2] \}$$

$$P(2, 2) = \alpha_1^2 \{ [1 - (1 - P)^2 (1 - p)^2] / [1 - (1 - P)(1 - p)^2] \}$$

where

$$\alpha_1 = \frac{p}{1 - (1 - P)(1 - p)}$$

The general expression for $L = 1$ when q_h is given by Eq. (2) is

$$P(i, m) = \sum_{j=0}^{i} \binom{m}{i} \binom{i}{j} \frac{(-1)^j P(1-p)^{m+j-i}}{1 - (1 - P)(1 - p)^{m+j-i}} \quad . \tag{4}$$

It is shown in Appendix B that as $T \to 0$ so that $L \to \infty$,

$$P(i, m) \to P_I(i, m) \qquad \text{where}$$

$$P_I(i, m) = \binom{m}{i} \alpha^i (1 - \alpha)^{m-i} \tag{5}$$

and

$$\alpha = \lim_{L \to \infty} P(1, 1) \quad .$$

The form of Eq. (5) implies that as $L \to \infty$ the user pairs become independent simce $\binom{m}{i} \alpha^i (1 - \alpha)^{m-i}$ is the probability that i of m independent users are transmitting if each transmits with probability α.

If decisions to transmit are made in a periodic rather than a pseudo-random manner

$$q_h = \begin{cases} 1 & h = H \\ 0 & h \neq H \end{cases}$$

where H is some integer. Eq (3) then reduces, for arbitrary L, to

$$P(i, m) = \binom{m}{i} \alpha_p^i (1 - \alpha_p)^{m-i}$$

where $\alpha_p = P(1, 1)$ so the users are independent.

In summary, pseudorandom frequency hopping introduces a dependence among the m user pairs in a group. This dependence is not present if the group uses the satellite in a periodic rather than a pseudorandom manner. If the number of transmissions, L, required to send a message is sufficiently large, this dependence can be made arbitrarily small. As will be seen from the discussion and examples of the following section, this dependence is detrimental to system operation in cases of practical interest.

Effect of Dependence on System Behavior

In terms of evaluating system behavior two previously defined quantities are of particular interest. They are the probability of an overload occurring,

$$Q(k+1, m) = \sum_{i=k+1}^{m} P(i, m) \quad ,$$

and the fraction of the information which is lost,

$$F_L(k+1, m) = \sum_{i=k+1}^{m} i\,P(i, m) \Big/ \sum_{i=0}^{m} i\,P(i, m) \quad .$$

These quantities can be computed for any values of p, P, m, k, and L from Eqs.(2) and (3) or from Eq. (4) for $L = 1$. The graphs of Examples 1 and 2 show the behavior of these quantities for various values of the parameters when $m = 2$ and $m = 3$.

Example 1

In Figure 1 the quantity $Q(2, 2)$, which is the probability of overload when $m = 2$ and $k = 1$, is plotted as a function of $\alpha = P(1, 1)$. The parameter P is chosen to be $1/256$ and curves for $L = 1$ and $L = \infty$ are shown. The parameter p is varied from 0 to 1 so that α varies from 0 to 1. It can be shown that for a fixed value of α, $Q(2, 2)$ decreases monotonically as L increases so that $L = 1$ and $L = \infty$ represent the extreme cases. The value of $Q(2, 2)$ does not change appreciably as P varies from 0.1 to 0 as long as p

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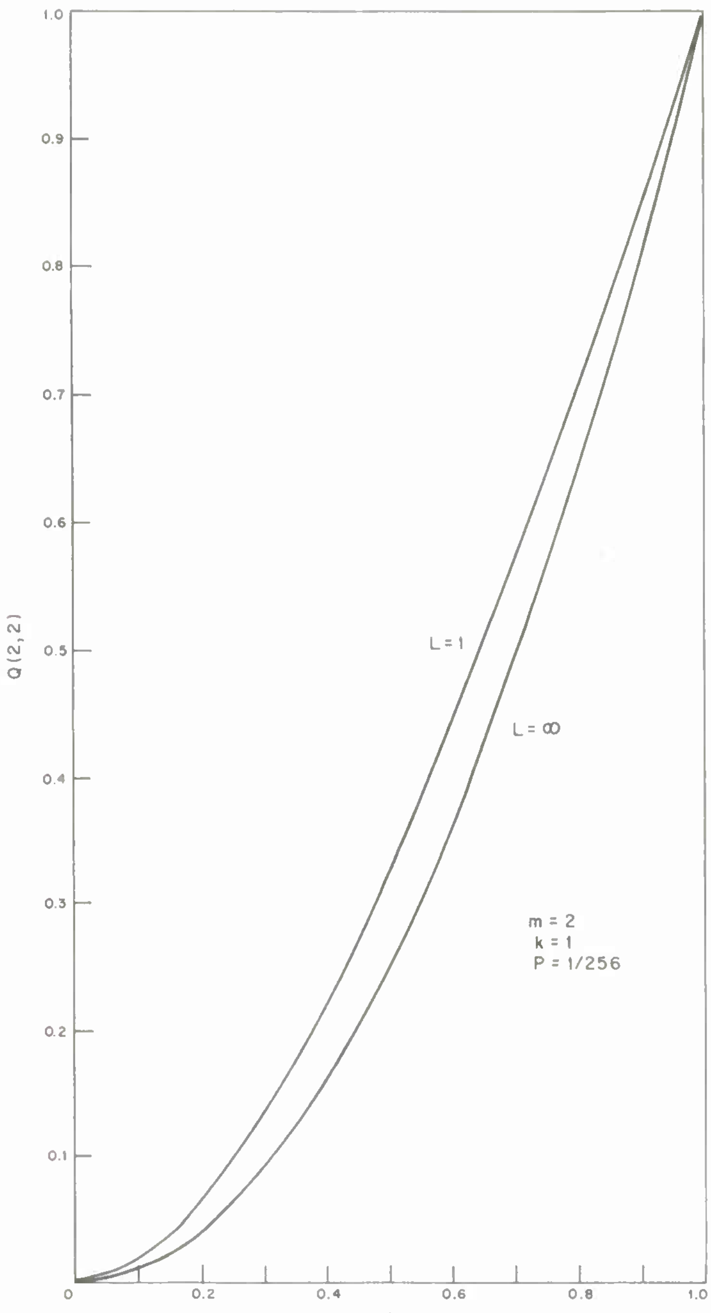

Fig. 1 Q(2, 2) vs α.

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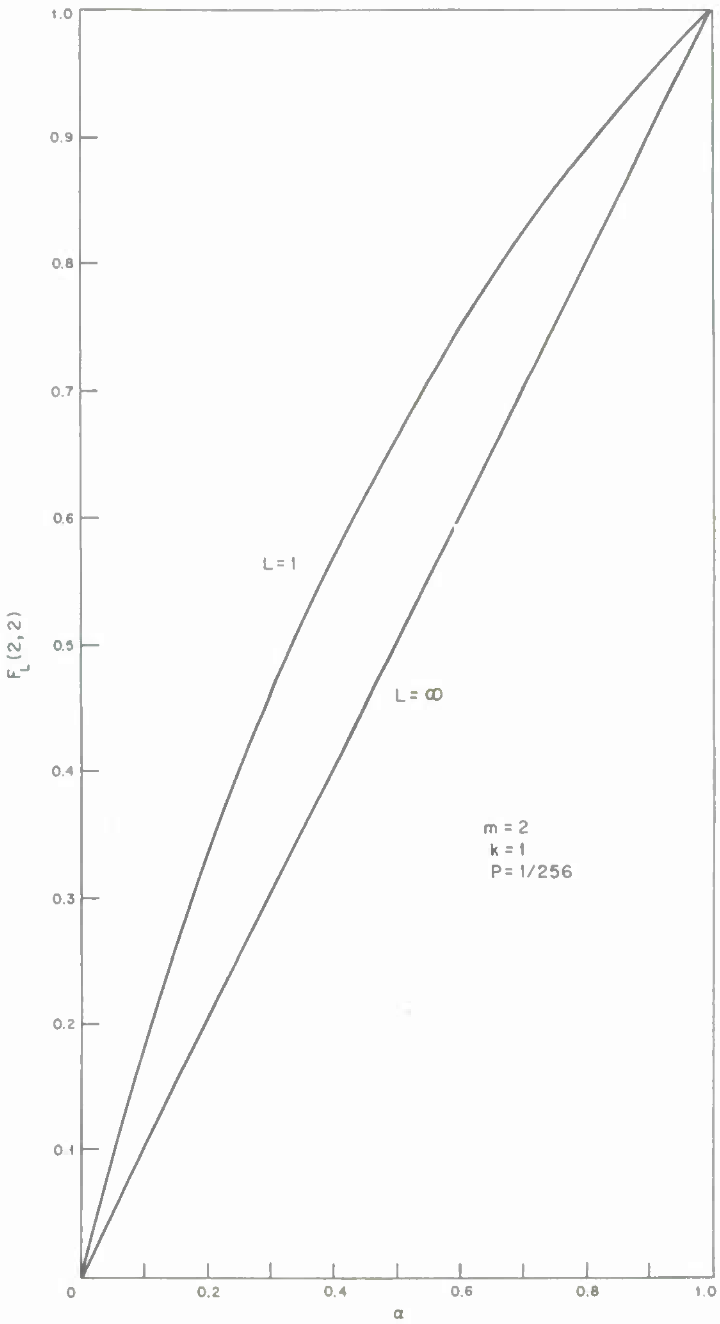

Fig. 2 $F_L(2, 2)$ vs α.

is adjusted to keep α invariant. Therefore, Figure 1 applies to a good approximation, for $0 < P < 0.1$. Figure 2 shows $F_L(2, 2)$ for the same parameters. For a fixed value of α, $F_L(2, 2)$ decreases monotonically as L increases, and $F_L(2, 2)$ does not change appreciably as P varies from 0.1 to 0. This follows because $F_L(2, 2) = Q(2, 2)/\alpha$.

Example 2

Figure 3 shows $Q(2, 3)$, Figure 4 shows $F_L(2, 3)$, Figure 5 shows $Q(3, 3)$, and Figure 6 shows $F_L(3, 3)$ which are the quantities which determine system performance when $m = 3$ and $k = 2$ and 3, respectively. In all cases curves are shown for $L = 1$ and $L = \infty$ and it can be shown that for $1 < L < \infty$ the curve for each of the four quantities lies between the respective curves for $L = 1$ and $L = \infty$. The figures are plotted for $P = 1/256$ but apply, to a good approximation, whenever $0 < P < 0.1$.

When $m >> 1$ it appears that in the cases of practical interest k should satisfy the relation $m > k >> 1$ because if $k \approx 1$ either $P(0, m)$ or $Q(k + 1, m)$ will be large and it is desirable that both be small. The case when $m > k >> 1$ and $Q(k + 1, m) << 1$ is analyzed below.

From Eq. (3) for arbitrary q_h and L the mean of the number of active user pairs i is

$$E[i] = \sum_{i=0}^{m} i\,P(i, m) = m\,P(1, 1) \quad .$$

The variance of i is

$$\mathrm{Var}(n) = E[(i - E[i])^2] = \sum_{i=0}^{m} [i - mP(1, 1)]^2\,P(i, m)$$

$$= m(m - 1)\,P(2, 2) + m\,P(1, 1)\,[1 - m\,P(1, 1)] \quad .$$

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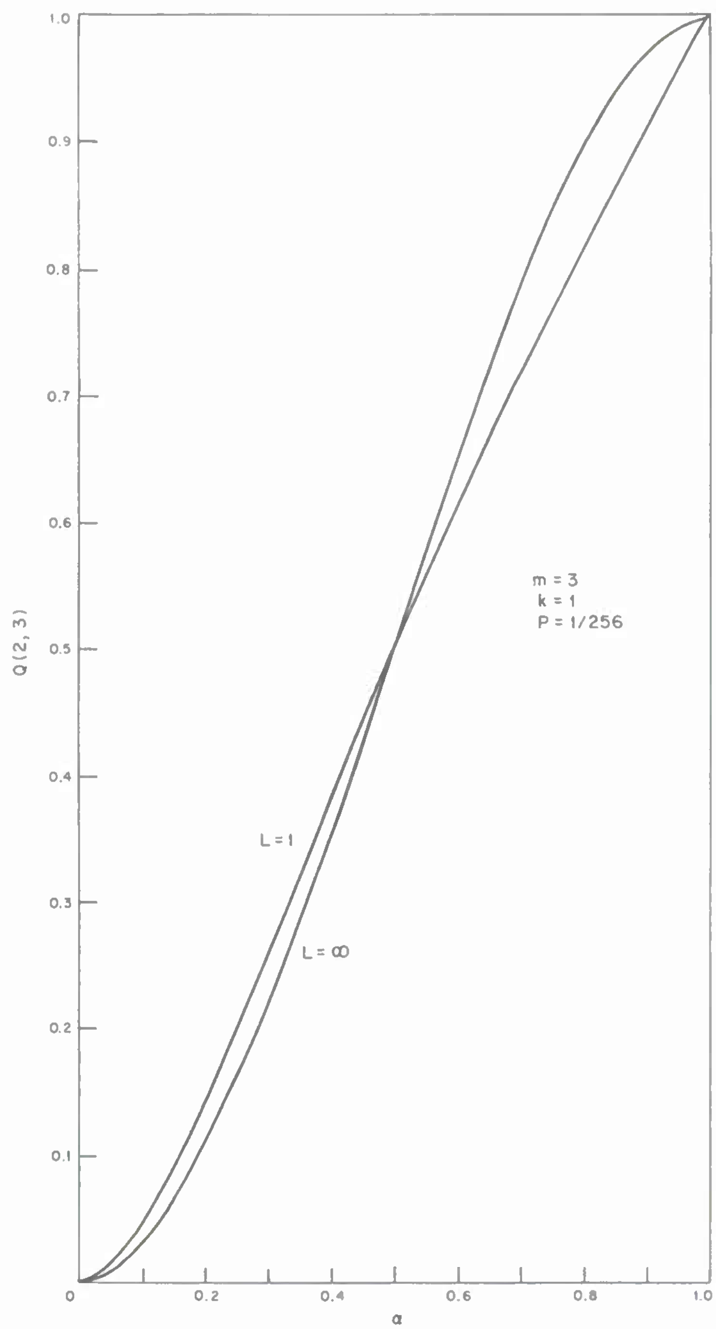

Fig. 3 Q(2, 3) vs α.

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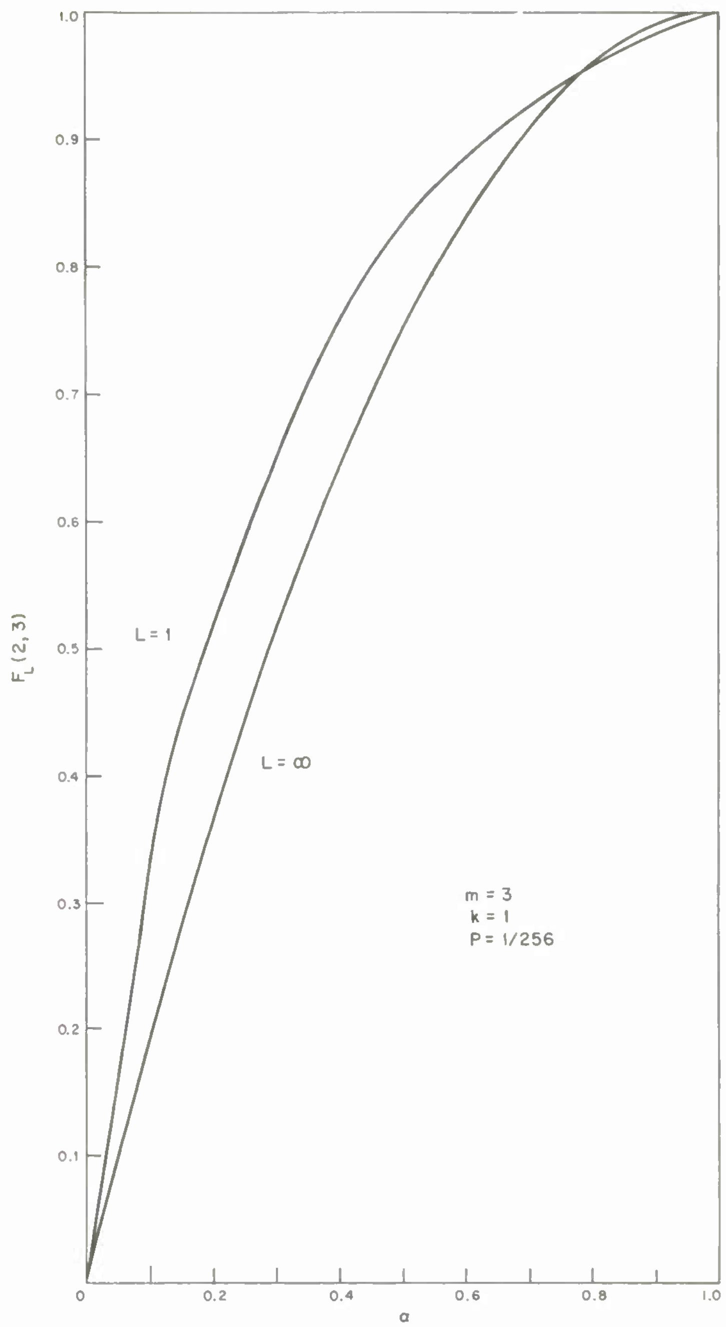

Fig. 4 $F_L(2,3)$ vs α.

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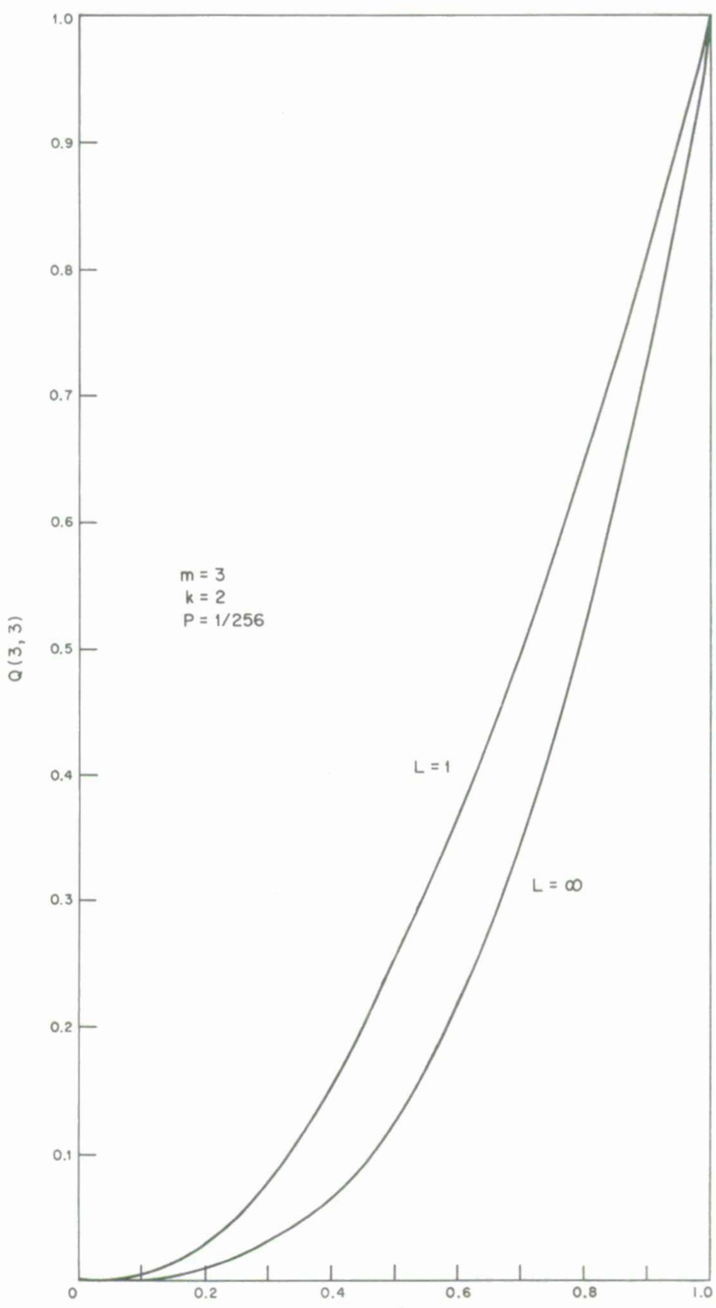

Fig. 5 Q(3, 3) vs α.

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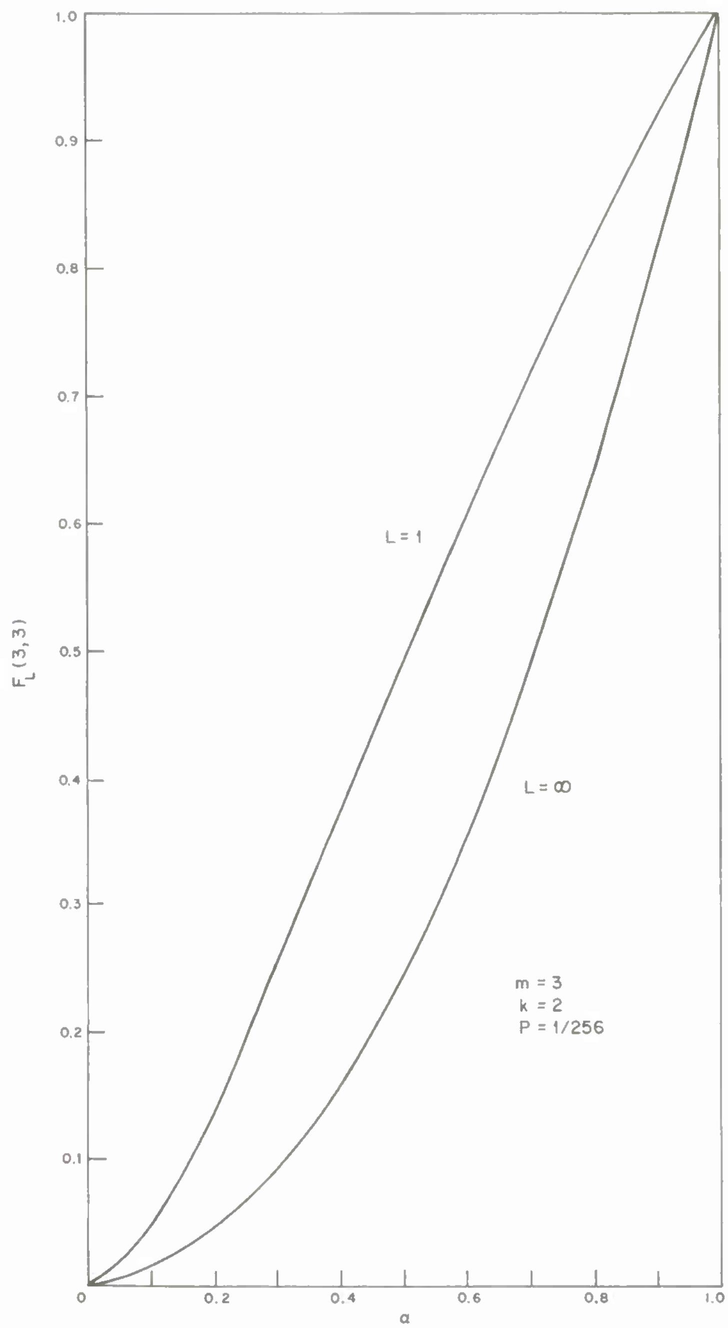

Fig. 6 $F_L(3, 3)$ vs α.

Two typical cases, A and B. of the distribution $P(i, m)$ are shown in Figure 7. The quantity $\sigma/\bar{i} = \sqrt{Var(i)/(E[i])^2}$ is larger for case A than for case B. For case A, $Q(k + 1, m)$ and $F_L(k + 1, m)$ are larger than for case B. The quantity $\sigma/\bar{i}$ is a rough measure of system performance. It is not as precise as $Q(k + 1, m)$ and $F_L(k + 1, m)$ but it can be computed more easily and presented much more compactly for arbitrary m than can these two quantities. Figure 8 shows $\sigma/\bar{i}$ as a function of $\alpha = P(1, 1)$ for various values of m and for $L = 1$ and $L = \infty$. It can be shown that, for a fixed value of α, as L increases monotonically from 1 to ∞, $\sigma/\bar{i}$ decreases monotonically. The curves of Figure 8 are plotted for $P = 1/128$ but they apply, to a good approximation, when $0 < P < 0.1$.

In all of the examples of this section the curves for $L = \infty$ can also be interpreted as being curves for periodic frequency hopping. Numerous values of $Q(k + 1, m)$, $F_L(k + 1, m)$, and $\sigma/\bar{i}$ calculated from the equations of this report and shown in the figures of this section have been checked by digital computer simulation. The values obtained from the calculations and from the simulations are essentially the same.

Discussion of Results and Conclusions

In the examples of the previous section and probably in any example, system performance as measured by $Q(k + 1, m)$ and $F_L(k + 1, m)$ is theoretically improved, when the system is not heavily overloaded, by increasing L. There are numerous ways in which L can be increased. The inter-hop time T can be decreased, the data rate of each user can be decreased thus causing the allowable number of simultaneous users k to be increased, and longer (larger number of bits) but fewer messages can be sent by each user.

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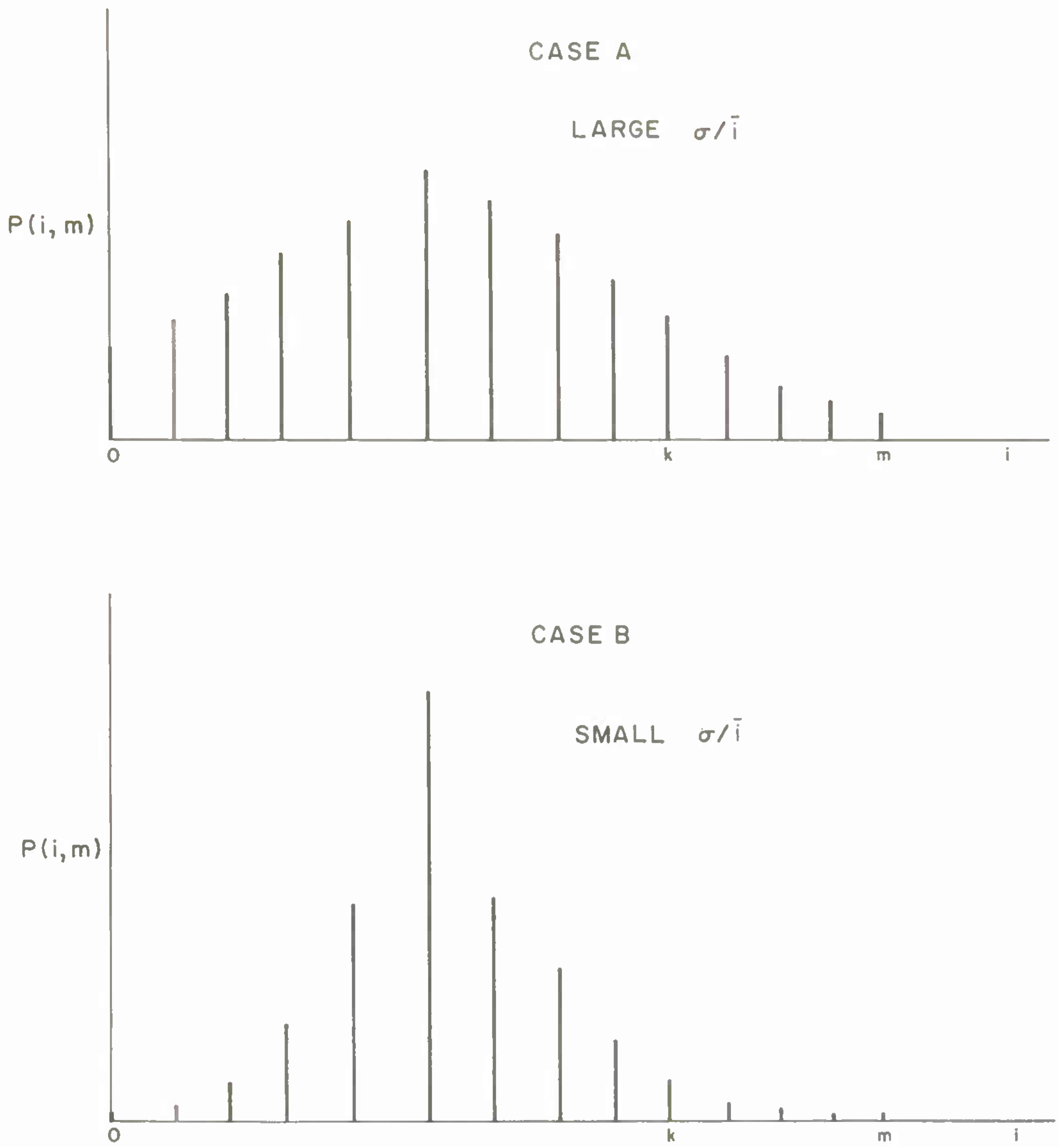

Fig. 7 Two Cases: Large and Small $\sigma/\bar{i}$.

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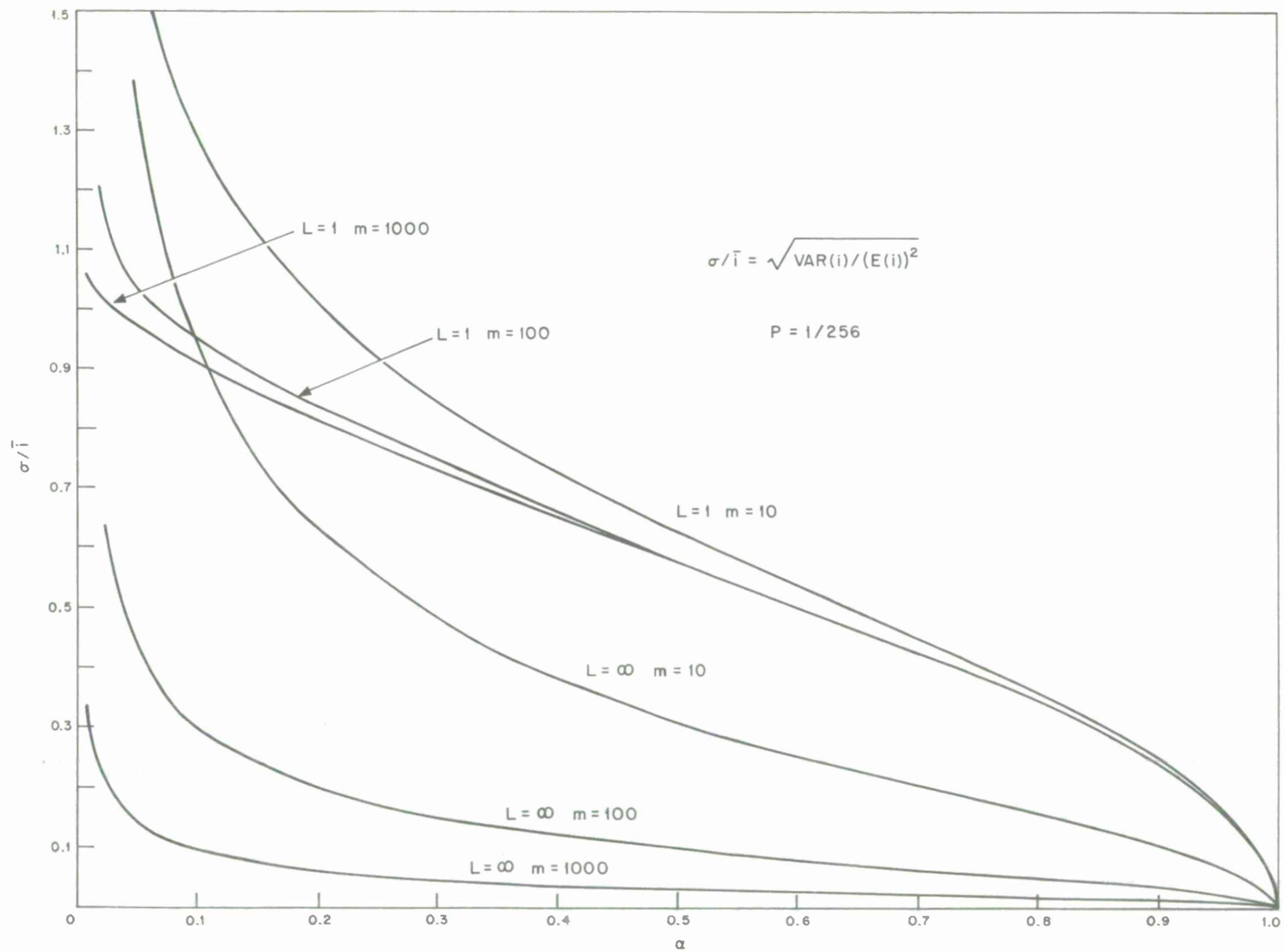

Fig. 8 $\sigma/\bar{i}$ vs α for $m = 10$, $m = 100$ and $m = 1000$.

However, there are several disadvantages associated with large L which do not appear in the preceding idealized analysis. The case $L = 1$ is inherently the simplest case and should require less equipment in the users' terminals. Due to the lack of perfect synchronization of the users and satellite and due to switching transients, the extremes of each time interval cannot be used for transmissions. This amount of time, t_g, which must be essentially wasted each time the satellite frequency hops is a function of the complexity of the user terminals and satellite borne equipment. As T decreases, the fraction of time that the satellite repeater can be used, $(T - t_g)/T$, decreases for fixed equipment complexity which implies constant t_g. A good choice of the number of users in a group, m, and the allowable number of simultaneously transmitting users, k, depends on many factors. In particular it is desirable to choose m and k so that $Q(k + 1, m)$ and $F_L(k + 1, m)$ are small rather than so that they are nearly equal to what they would be if the users were independent. The length (number of bits) of a message is usually determined by other considerations. Finally, another measure of system performance may be more realistic than $F_L(k, m)$, the fraction of the information lost. If a message requires L transmissions the message may be worthless to the receiver when some number $0 < n < L$ of the transmissions are lost. Although the number of users transmitting, at successive times when the group transmits, is highly correlated when $L > 1$, it is possible to have only n of the L transmissions lost. A meaningful measure of system performance in this case is $F_L'(k + 1, m)$, the fraction of the messages which are lost. In general,

$$F_L'(k + 1, m) \geq F_L(k + 1, m)$$

and the equality holds for $L = 1$. Although $F_L(k + 1, m)$ decreases with increasing L, $F_L'(k + 1, m)$ may not.

The best choice of the parameters m, k, L, p, and P is a complex problem in which many factors must be considered. Another report will be directed toward examining the trade-offs in the problem. Particular choices of the parameters can be compared by using the expressions for $Q(k + 1, m)$ and $F_L(k + 1, m)$ of the report. The degradation in system performance represented by increases in $Q(k + 1, m)$ and $F_L(k + 1, m)$ caused by frequency hopping in a pseudorandom rather than a periodic manner can be quantitatively evaluated.

APPENDIX A

In this section it will be verified that Eq. (3) is correct. From the definition of conditional probability

$$P(i, m) = \sum_{h=0}^{\infty} A(i, m, h)\, q_h$$

where $A(i, m, h)$ equals the probability that i of the m user pairs are transmitting at a time when the group transmits conditional upon the event that h times at which other groups transmit separate this time and the previous time that the group transmitted. Therefore it suffices to show that

$$A(i, m, h) = \binom{m}{i} \left[L - L(1-p)^{h+1}\right]^i \left[(1-p)^{h+1}\right]^{m-i} / \left[L - (L-1)(1-p)^{h+1}\right]^m \quad . \tag{A. 1}$$

Consider the case $L = 1$ in which the numbers of pairs transmitting at successive times are independent. When there are h time intervals between successive transmissions of the group, there are $h + 1$ possible times for a message to arrive. The probability that no message arrives for a given pair to transmit is $(1-p)^{h+1}$ since arrivals at successive times are independent. The probability that a message arrives is $1 - (1-p)^{h+1}$. Since arrivals for different pairs are independent,

$$A(i, m, h) = \binom{m}{i} \left[1 - (1-p)^{h+1}\right]^i \left[(1-p)^{h+1}\right]^{m-i}$$

and Eq. (A. 1) is verified for $L = 1$.

For $L > 1$ the numbers of pairs transmitting at successive times are not independent so it is useful to employ a different argument. The state of the m pairs at any discrete time can be represented by a vector of integers, $S = (I_1, I_2, \ldots, I_m)$ where each I ranges from zero to L. If $I_i = 0$ the ith user pair has no message to transmit and if $I_i = k$ where $0 < k \leq L$ the ith user pair has a fraction k/L of a message remaining to be transmitted. Future values of S depend only on its present value and on future arrivals and transmissions of messages. Future arrivals and the random element in future transmissions are independent of the present value of S so this vector represents the state of the users in the Markov process sense.

In general, there is a number of states in which i of the users are simultaneously transmitting. By enumerating the possibilities it is determined that this number is $\binom{m}{i} (L)^i$. It can be shown that every state, in which there is the same number of active users, is equally likely. Although it is not obvious, this conclusion follows from the symmetry induced by the fact that all users have identical statistical descriptions and from the fact that whenever I_k changes from 0 to L it subsequently changes to $L-1$, $L-2, \ldots, 1$.

Let γ_c equal the probability that i particular elements of S equal L when the group next transmits given that no messages remain to be transmitted at the present time, which is the time that the group completes a transmission and which is just prior to when possible arrivals can occur. Let γ equal the probability that no messages remain to be transmitted at the present time. Since arrivals occur for i particular users,

$$\gamma_c \gamma = [1 - (1-p)^{h+1}]^i \, [(1-p)^{h+1}]^{m-1} \gamma \qquad .$$

The quantity $\gamma_c\gamma$ is the probability that one state which leads to i simultaneous users occurs. Since there are $\binom{m}{i}(L)^i$ of these states and they are equally likely,

$$A(i, m, h) = \binom{m}{i}\left[L - L(1-p)^{h+1}\right]^i \left[(1-p)^{h+1}\right]^{m-i}\gamma$$

Since $\sum_{i=0}^{m} A(i, m, h) = 1$,

$$\gamma = 1/\left[L - (L-1)(1-p)^{h+1}\right]^m$$

so Eq. (A. 1) follows.

APPENDIX B

In this section it will be shown that

$$\lim_{T \to 0} P(i, m) = \binom{m}{i} \alpha^{i}(1 - \alpha)^{m-i}$$

where $\alpha = \lim_{T \to 0} P(1, 1)$ and $P(i, m)$ is given by Eq. (3). Since L must remain an integer it is no restriction to let $L = L_0 n$ and $T = T_0/n$ where n and L_0 are positive integers and $n \to \infty$. The probability of no message arrival at a given user pair in $h + 1$ discrete times, $(1 - p)^{h+1}$, must be adjusted to depend on n so that the mathematical model remains meaningful for all n. This adjustment can be made in any of several ways, all of which yield the same result as $n \to \infty$. One way to accomplish this adjustment is to require that the probability of no arrival occurring in a given time between transmissions of the group remain invariant as n varies. In an interval of length hT_o seconds there are $1 + h/n$ discrete times when a message can arrive (counting both end points) since a discrete arrival time occurs every T seconds. Therefore, $(1 - p)^{h+1}$ is adjusted to be $(1 - p)^{h/n+1}$. At an idle user a message arrives with probability p every discrete time so the number of transmissions required to send the expected amount of message which arrives at a given time is $pL_0 n$. To make this number independent of n we require that $p = p_0/n$. This model provides a meaningful representation of message arrivals which is valid for all positive n and which agrees with the model of previous sections when $n = 1$. Eq. (3) becomes

$$P(i,m) = \binom{m}{i} \sum_{h=0}^{\infty} q_h \{ [nL_0 - nL_0(1-p_0/n)^{h/n+1}]^i \cdot$$

$$\cdot [(1-p_0/n)^{h/n+1}]^{m-i}\} / \{ [nL_0 - (nL_0 - 1)(1-P_0/n)^{h/n+1}]^m\}.$$

Since the convergence properties of the series justifies interchanging the order of the summation and limit and since

$$\lim_{n\to\infty} [nL_0 - nL_0(1-p_0/n)^{h/n+1}] = L_0 p_0 \quad ,$$

$$\lim_{n\to\infty} [(1-p_0/n)^{h/n+1}] = 1$$

and

$$\lim_{n\to\infty} [nL_0 - (nL_0 - 1)(1-p_0/n)^{h/n+1}] = 1 + L_0 p_0 \quad ,$$

$$\lim_{n\to\infty} P(i,m) = \binom{m}{i} [L_0p_0/(1+L_0p_0)]^i [1/(1+L_0p_0)]^{m-i}$$

because $\sum_{h=0}^{\infty} q_h = 1$. This is the result to be shown since as $n \to \infty$, $T \to 0$.

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REFERENCE

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13. ABSTRACT

When the up-link frequency band repeated by a communications satellite is hopped in a periodic manner, when users are divided so that a group of them can transmit only when a particular frequency band is repeated, and when message arrivals for one user are independent of message arrivals for any other user, the users in a group are independent. When the frequency hopping is done in a pseudorandom rather than a periodic manner, the users are dependent. The effect of the dependence is, in practical cases, to increase the probability of there being a large number of simultaneous users which increases the probability of system overload. General expressions for the probability of system overload and for the fraction of the information lost due to overload are obtained. These expressions illustrate the effect of the dependence induced by the pseudorandom frequency hopping. Examples are presented.

14. KEY WORDS

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